Graphs with large obstacle numbers*

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Abstract. Motivated by questions in computer vision and sensor networks, Alpert et al. [3] introduced the following definitions. Given a graph G, an *obstacle representation* of G is a set of points in the plane representing the vertices of G, together with a set of connected obstacles such that two vertices of G are joined by an edge if an only if the corresponding points can be connected by a segment which avoids all obstacles. The *obstacle number* of G is the minimum number of obstacles in an obstacle representation of G. It was shown in [3] that there exist graphs of n vertices with obstacle number at least $\Omega(\sqrt{\log n})$. We use extremal graph theoretic tools to show that (1) there exist graphs of n vertices with obstacle number at least $\Omega(n/\log^2 n)$, and (2) the total number of graphs on n vertices with bounded obstacle number is at most $2^{o(n^2)}$. Better results are proved if we are allowed to use only *convex* obstacles or polygonal obstacles with a small number of sides.

1 Introduction

Consider a set P of points in the plane and a set of closed polygonal obstacles whose vertices together with the points in P are in *general position*, that is, no *three* of them are on a line. The corresponding *visibility graph* has P as its vertex set, two points $p,q \in P$ being connected by an edge if and only if the segment pq does not meet any of the obstacles. Visibility graphs are extensively studied and used in computational geometry, robot motion planning, computer vision, sensor networks, etc.; see [5], [15], [20], [21], [31].

Recently, Alpert, Koch, and Laison [3] introduced an interesting new parameter of graphs, closely related to visibility graphs. Given a graph G, we say that a set of points and a set of polygonal obstacles as above constitute an *obstacle representation* of G, if

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the corresponding visibility graph is isomorphic to G. A representation with h obstacles is also called an h-obstacle representation. The smallest number of obstacles in an obstacle representation of G is called the *obstacle number* of G and is denoted by $\mathrm{obs}(G)$. If we are allowed to use only *convex* obstacles, then the corresponding parameter $\mathrm{obs}_c(G)$ is called the *convex obstacle number* of G. Of course, we have $\mathrm{obs}(G) \le \mathrm{obs}_c(G)$ for every G, but the two parameters can be very far apart.

A special instance of the obstacle problem has received a lot of attention, due to its connection to the Szemerédi-Trotter theorem on incidences between points and lines [28], [27], and other classical problems in incidence geometry [23]. It is an exciting open problem to decide whether the obstacle number of $\overline{K}_n$, the empty graph on n vertices, is $O(n)$ if the obstacles must be *points*. The best known upper bound is $n2^{O(\sqrt{\log n})}$; see Pach [22], Dumitrescu et al. [7], Matoušek [18], and Aloupis et al. [2].

Alpert et al. [3] constructed a bipartite graph G_1 and a split graph (a graph consisting of a clique and an independent set with possible edges between them) G_2 with obstacle number at least *two*. In Section 4, we complement their examples with a third one:

Theorem 1. *There is a graph G_3 that consists of two cliques with edges between them and satisfies* $\mathrm{obs}(G_3) \ge 2$.

Consequently, no graph of obstacle number *one* has an induced subgraph isomorphic to G_1, G_2, or G_3. The choice of these forbidden graphs may appear somewhat capricious at first glance. In Section 2, we will see that this set of graphs allows us to utilize some extremal graph theoretic tools developed by Erdős, Kleitman, Rothschild, Frankl, Rödl, Prömel, Steger, Bollobás, Thomason and others. They yield that the number of graphs with n vertices and bounded obstacle number is very small, compared to the total number of labeled graphs, which is $2^{\binom{n}{2}}$. More precisely, we obtain

Corollary 1. *For any fixed positive integer h, the number of graphs on n (labeled) vertices with obstacle number at most h is at most $2^{o(n^2)}$.*

Alpert et al. [3] raised the question whether there exist *bipartite* graphs with arbitrarily large obstacle number? Since the number of bipartite graphs with n labeled vertices is $\Omega(2^{n^2/4})$, it follows directly from Corollary 1 that the answer is yes.

Corollary 2. *For any fixed positive integer h, there exist bipartite graphs with obstacle number at least h.*

For every sufficiently large n, Alpert et al. constructed a graph with n vertices with obstacle number at least $\Omega\left(\sqrt{\log n}\right)$. We also show in Section 2 how Theorem 1, combined with a result by Erdős and Hajnal [8], implies the existence of graphs with much larger obstacle numbers.

Corollary 3. *For every $\varepsilon > 0$, there exists an integer $n_0 = n_0(\varepsilon)$ such that for all $n \ge n_0$, there are graphs G on n vertices such that their obstacle numbers satisfy*

$$\mathrm{obs}(G) \ge \Omega\left(n^{1-\varepsilon}\right) .$$

It turns out that for the proof of Corollary 3, in the place of Theorem 1 we can use the much simpler fact that there are graphs with obstacle number greater than *one*. In Section 3, we improve on the last two corollaries, using some estimates on the number of different *order types* of n points in the Euclidean plane, discovered by Goodman and Pollack [16], [17] (see also Alon [1]). We establish the following results.

Theorem 2. *For any fixed positive integer h, the number of graphs on n (labeled) vertices with obstacle number at most h is at most*

$$2^{O(hn\log^2 n)} \quad .$$

Theorem 3. *For every n, there exist graphs G on n vertices with obstacle numbers*

$$\mathrm{obs}(G) \geq \Omega\left(n/\log^2 n\right) \quad .$$

Note that Theorem 3 directly follows from Theorem 2. Indeed, since the total number of (labeled) graphs with n vertices is $2^{\Omega(n^2)}$, as long as $2^{O(hn\log^2 n)}$ is smaller than this quantity, there is a graph with obstacle number larger than h.

We prove a slightly better bound for convex obstacle numbers.

Theorem 4. *For every n, there exist graphs G on n vertices with convex obstacle numbers*

$$\mathrm{obs}_c(G) \geq \Omega\left(n/\log n\right) \quad .$$

If we only allow segment obstacles, we get an even better bound. Following Alpert et al., we define the *segment obstacle number* $\mathrm{obs}_s(G)$ of a graph G as the minimal number of obstacles in an obstacle representation of G, in which each obstacle is a straight-line segment.

Theorem 5. *For every n, there exist graphs G on n vertices with segment obstacle numbers*

$$\mathrm{obs}_s(G) \geq \Omega\left(n^2/\log n\right) \quad .$$

In Section 4, we prove Theorem 1.

In the last section, we make some concluding remarks. In particular, we answer a question of Alpert et al. [3] by showing that for every positive integer h, there exists a graph with obstacle number *precisely* h. We also discuss possible extensions of the above notions to higher dimensions.

Given any placement (embedding) of the vertices of G in general position in the plane, a *drawing* of G consists of the image of the embedding and the set of *open segments* connecting all pairs of points that correspond to the edges of G. If there is no danger of confusion, we make no notational difference between the vertices of G and the corresponding points, and between the pairs uv and the corresponding open segments. The complement of the set of all points that correspond to a vertex or belong to at least one edge of G falls into connected components. These components are called the *faces* of the drawing. Notice that if G has an obstacle representation with a particular placement of its vertex set, then

(1) each obstacle must lie entirely in one face of the drawing, and

(2) each non-edge of G must be blocked by at least one of the obstacles.

2 Hereditary properties, universal graphs, and applications

The aim of this section is to review some results in extremal graph theory and then to apply them to establish Corollaries 1 and 3.

In 1985, Erdős, Kleitman, and Rothschild [11] proved that, as n tends to infinity, the number of all K_ℓ-free graphs on n vertices is asymptotically equal to the number of $(\ell-1)$-partite graphs with n vertices with as equal vertex classes as possible. This result was soon generalized to graphs that do not contain some fixed (not necessarily induced) subgraph H [10].

Analogous questions based on the *induced* subgraph relation were investigated in [24], [26], and [25]. If a graph G does not contain an induced subgraph isomorphic to a fixed graph H, then the same is true for every induced subgraph of G. Therefore, this property is called *hereditary*. In order to formulate an Erdős-Kleitman-Rothschild type theorem valid for any hereditary graph property, we need some definitions and notations.

In notation, we do not distinguish between a graph property $\mathscr{P}$ and the set of all graphs that satisfy this property. In the same spirit, the set of all graphs on n labeled vertices, which satisfy property $\mathscr{P}$, is denoted by $\mathscr{P}^n$.

A graph is (r,s)-colorable if its vertex set can be partitioned into r blocks out of which s are cliques and every remaining block is an independent set. Let $\mathscr{C}(r,s)$ denote the set of all (r,s)-colorable graphs. A graph property which holds for all graphs is called *trivial*. Given any nontrivial hereditary graph property $\mathscr{P}$, define its *coloring number* as

$$r(\mathscr{P}) = \max\{r \mid \exists s : \mathscr{C}(r,s) \subseteq \mathscr{P}\} \ .$$

Since $r(\mathscr{P})$ is bounded from above by the number of vertices of any graph that does not satisfy $\mathscr{P}$, the parameter $r(\mathscr{P})$ exists and it is at least 1.

Theorem 6 (Bollobás, Thomason [6]). *For any nontrivial hereditary graph property $\mathscr{P}$, we have*

$$|\mathscr{P}^n| = 2^{\left(1-\frac{1}{r(\mathscr{P})}+o(1)\right)\binom{n}{2}} \ .$$

Notice that if for some value r there is no s such that $\mathscr{C}(r,s) \subseteq \mathscr{P}$, then for every $r' > r$ there is no s for which $\mathscr{C}(r',s) \subseteq \mathscr{P}$. If there are $(2,0)$-colorable, $(2,1)$-colorable, and $(2,2)$-colorable graphs *none* of which is in $\mathscr{P}$, then by the preceding observations, $r(\mathscr{P}) = 1$. In that case, by Theorem 6, we have $|\mathscr{P}^n| = 2^{o(n^2)}$.

The familiar term for a $(2,0)$-colorable graph is bipartite. A $(2,1)$-colorable graph consists of a clique and an independent set, possibly with edges running between them; such a graph is often called a *split graph* [13], [30]. A $(2,2)$-colorable graph consists of two cliques, possibly with edges running between them—its complement is bipartite.

Proof of Corollary 1. We apply Theorem 6 to the hereditary property that a graph admits a 1-obstacle representation. The graphs G_1, G_2, and G_3 defined in the Introduction are $(2,0)$-, $(2,1)$-, and $(2,2)$-colorable. Thus, in view of the fact that, according to Alpert et al. and Theorem 1, none of them admits a 1-obstacle representation, we can conclude that the number of all graphs on n (labeled) vertices with obstacle number at most 1 is $2^{o(n^2)}$. In other words, Corollary 1 holds for $h = 1$.

For every fixed $h > 1$, consider a graph G on the vertex set $[n]$, which permits an h-obstacle representation on an n-element point set P in general position, with obstacles $O_1, \dots, O_h$. Obviously, $E(G)$, the edge set of G, can be obtained as $\cap_{i=1}^{h} E(G_i)$, for suitable graphs G_i with obstacle number 1. Indeed, we can choose G_i to be the visibility graph of P in the presence of a single obstacle O_i $(i = 1, \dots, h)$. Therefore, the total number of labeled graphs on $[n]$ with obstacle number h can be bounded from above by the h-th power of the number of graphs with obstacle number 1. This completes the proof of Corollary 1. □

Let G be a graph on n vertices and let k be a positive integer. We say that G is k-*universal* if it contains every graph on k vertices as an induced subgraph. Let $\hom(G)$ denote the maximum of the size of the largest independent set of vertices and the size of the largest complete subgraph in G. According to the quantitative form of Ramsey's theorem, due to Erdős and Szekeres [12], $\hom(G)$ is at least roughly $\frac{1}{2}\log n$. (In the sequel, all logarithms are taken modulo 2.)

In order to prove Corollary 3, we need the following result, which shows that if G avoids at least one induced subgraph with k vertices, for some $k \ll \log n$, then the Erdős-Szekeres bound on $\hom(G)$ can be substantially improved.

Theorem 7 (Erdős, Hajnal [8]). *For any fixed positive integer t, there is an $n_0 = n_0(t)$ with the following property. Given any graph G on $n > n_0$ vertices and any integer $k < 2^{c\sqrt{\log n}/t}$, either G is t-universal or we have $\hom(G) \geq k$. (Here $c > 0$ is a suitable constant.)*

Proof of Corollary 3. For the sake of clarity of the presentation, we systematically omit all floor and ceiling functions wherever they are not essential. Let H be a graph of t vertices that does not admit a 1-obstacle representation. Fix any $0 < \varepsilon < 1$, and choose an integer $N \geq n_0$, that satisfies the inequality

$$2^{c\sqrt{\varepsilon \log N}/t} > 2\log N \quad , \tag{1}$$

where c, n_0 are constants that appear in the previous theorem.

For any $n \geq N$, we set $m = n^{1-\varepsilon}$. According to a theorem of Erdős [9], there exists a graph G with n vertices such that

$$\hom(G) < 2\log n < 2^{c\sqrt{\log(n/m)}/t} \quad .$$

Consider an obstacle representation of G with the smallest number h of obstacles. Suppose without loss of generality that in our coordinate system all points of G have different x-coordinates. By vertical lines, partition the plane into m strips, each containing n/m points. Let G_i denote the subgraph of G induced by the vertices lying in the i-th strip $(1 \leq i \leq m)$.

Obviously, we have

$$\hom(G_i) \leq \hom(G) < 2^{c\sqrt{\log(n/m)}/t} \quad ,$$

for every i. Hence, applying Theorem 7 to each G_i separately, we conclude that each must be t-universal. In particular, each G_i contains an induced subgraph isomorphic to

H. That is, we have $\text{obs}(G_i) > 1$ for every i, which means that each G_i requires at least *two* obstacles.

As was explained at the end of the Introduction, each obstacle must be contained in an interior or in the exterior face of the graph. Therefore, in an h-obstacle representation of G, each G_i must have at least one internal face that contains an obstacle, and there must be at least one additional obstacle (which may possibly contained in the interior face of every G_i). At any rate, we have $h > m = n^{1-\varepsilon}$, as required. □

3 Encoding graphs of low obstacle number

The aim of this section is to prove Theorems 2–5. The idea is to find a short encoding of the obstacle representations of graphs, and to use this to give an upper bound on the number of graphs with low obstacle number.

We need to review some simple facts from combinatorial geometry. Two sets of points, P_1 and P_2, in general position in the plane are said to have the same *order type* if there is a one to one correspondence between them with the property that the orientation of any triple in P_1 is the same as the orientation of the corresponding triple in P_2. Counting the number of different order types is a classical task, see e.g.

Theorem 8 (Goodman, Pollack [16]). *The number of different order types of n points in general position in the plane is* $2^{O(n\log n)}$.

Observe that the same upper bound holds for the number of different order types of *n labeled* points, because the number of different permutations of n points is $n! = 2^{O(n\log n)}$.

In a graph drawing, the *complexity* of a face is the number of line segment sides bordering it. The following result was proved by Arkin, Halperin, Kedem, Mitchell, and Naor (see Matoušek, Valtr [19] for its sharpness).

Theorem 9 (Arkin et al. [4]). *The complexity of a single face in a drawing of a graph with n vertices is at most* $O(n\log n)$.

Note that this bound does not depend of the number of edges of the graph.

Proof of Theorem 2. For any graph G with n vertices that admits an h-obstacle representation, fix such a representation. Consider the visibility graph G of the vertices in this representation. As explained at the end of the Introduction, every obstacle belongs to a single face in this drawing. In view of Theorem 9, the complexity of every face is $O(n\log n)$. Replacing each obstacle by a slightly shrunken copy of the face containing it, we can achieve that every obstacle *is* a polygonal region with $O(n\log n)$ sides.

Let S be the point sequence starting with the vertices of G, followed by the vertices of every obstacle in cyclic order, one entire obstacle after another. Let I be the set of the starting positions of the h obstacles in S. G is completely determined by the (labeled) order type of S, together with I. To see this, first observe that I tells us which pairs in S are pairs graph vertices and which correspond to a side of some polygon. Now, notice that a given segment uv among graph vertices is blocked if and only if it meets some side ab of some polygon, for which a necessary and sufficient condition is that the ordered triples uav, avb, vbu, and bua have the same orientation.

If the length of S is N, then the number of possibilities for I is at most $\binom{N}{h} \le N^h$. Since $N \le n + c_1 hn\log n$ for some absolute constant $c_1 > 0$, according to Theorem 8 and our comment that follows it, the number of graphs with obstacle number at most h is at most

$$N^h \cdot 2^{O(N\log N)} = 2^{O(N\log N)} < 2^{chn\log^2 n} \quad ,$$

for a suitable constant $c > 0$. This is a generous upper bound due to overcounting, and also because most pairs (S,I) do not encode obstacle representations. □

If the average number of sides an obstacle can have is small, then we obtain

Theorem 10. *The number of graphs admitting an obstacle representation with at most h obstacles, having a total of at most hs sides, is at most*

$$2^{O(n\log n + hs\log(hs))} \quad .$$

In particular, for segment obstacles ($s = 2$), Theorem 10 immediately implies Theorem 5. Indeed, as long as the bound in Theorem 10 is smaller than $2^{\binom{n}{2}}$, the total number of graphs on n labeled vertices, we can argue that there is a graph with segment obstacle number larger than h.

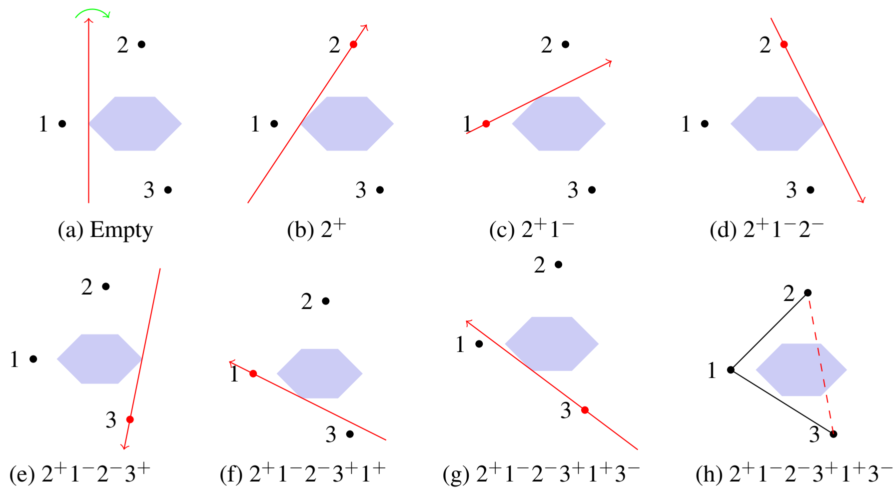

Fig. 1: Parts (a) to (g) show the construction of the sequence and (h) shows the visibilities. The arrow on the tangent line indicates the direction from the point of tangency in which we assign $+$ as a label to the vertex. The additional arrow in (a) indicates that the tangent line is rotated clockwise around the obstacle.

Proof of Theorem 4. As before, it is enough to bound the number of graphs that admit an obstacle representation with at most h convex obstacles. Let us fix such a graph G, together with a representation. Let V be the set of points representing the vertices, and let $O_1, \ldots, O_h$ be the convex obstacles. For any obstacle O_i, rotate an oriented tangent line ℓ along its boundary in the clockwise direction. We can assume without loss of generality that ℓ never passes through two points of V. Let us record the sequence of points met by ℓ. If $v \in V$ is met at the right side of ℓ, we add the symbol v^+ to the sequence, otherwise we add v^- (Fig. 1). When ℓ returns to its initial position, we stop. The resulting sequence consists of $2n$ characters. From this sequence, it is easy to reconstruct which pairs of vertices are visible in the presence of the single obstacle O_i. Observe that O_i blocks uv if and only if the subsequence induced on u and v has no consecutive pair with the same superscript. Hence, knowing these sequences for every obstacle O_i, completely determines the visibility graph G. The number of distinct sequences assigned to a single obstacle is at most $(2n)!$, so that the number of graphs with convex obstacle number at most h cannot exceed $((2n)!)^h/h! < (2n)^{2hn}$. As long as this number is smaller than $2^{\binom{n}{2}}$, there is a graph with convex obstacle number larger than h. □

4 Proof of Theorem 1

Let the graph G_3 consist of a clique of blue vertices $B = \{b_i \mid i \in [4]\}$, a clique of red vertices $R = \{r_A \mid A \subseteq [4]\}$, and additional edges between every b_i and every r_A with $i \in A$. We say that a polygon is *solid* if all its edges are edges in G_3. For three distinct points p, q, and r, we denote by $\angle pqr$ the union of the rays $\overrightarrow{qp}$ and $\overrightarrow{qr}$. For a point set P, we denote by $\text{conv}(P)$ the convex hull of P (the smallest convex set containing P).

Assume for contradiction that we are given a 1-obstacle representation of G_3. For a red vertex r_A, if there are points p and q such that $\angle p r_A q$ strictly separates $\{b_i \mid i \in A\}$ from the remaining blue vertices, we say that r_A is *innocent*. If some red vertex r_A is not innocent, two obstacles will be required due to $\{r_A\} \cup B$, a contradiction.

Case 1: B is not in convex position. Without loss of generality, b_4 is inside triangle $\Delta b_1 b_2 b_3$.

Subcase 1a: The obstacle is in $\text{conv}(B)$. Without loss of generality, the obstacle is inside $\Delta b_1 b_4 b_3$. Then $r_{\{1,4\}}$ is inside $\Delta b_1 b_4 b_3$, for the obstacle to block $b_2 r_{\{1,4\}}$ and $b_3 r_{\{1,4\}}$. Similarly, $r_{\{3,4\}}$ is inside $\Delta b_1 b_4 b_3$. For $r_{\{1,4\}}$ and $r_{\{3,4\}}$ to be innocent, the line through b_2 and b_4 separates $b_1 r_{\{3,4\}}$ from $b_3 r_{\{1,4\}}$. Without loss of generality, $r_{\{1,4\}}$ is inside $\Delta b_4 r_{\{3,4\}} b_3$. Since $b_1 r_{\{3,4\}}$ and $b_3 r_{\{1,4\}}$ are separated by the solid $\Delta b_4 r_{\{3,4\}} b_3$, two obstacles are needed, a contradiction.

Subcase 1b: The obstacle is outside of $\text{conv}(B)$. Hence $r_{\{1,2,3\}}$ is outside of $\text{conv}(B)$, and without loss of generality, in $\text{conv}(\angle b_1 b_4 b_3)$. Therefore, the obstacle is inside the convex quadrilateral $Q = b_1 b_4 b_3 r_{\{1,2,3\}}$. For $b_1 r_4$ and $b_3 r_4$ to be blocked, r_4 is inside Q. Then $\angle b_4 r_4 r_{\{1,2,3\}}$ separates $\text{conv}(Q)$ into two regions with solid boundaries that respectively contain $b_1 r_4$ and $b_3 r_4$. Therefore, two obstacles are needed, a contradiction.

Case 2: B is in convex position. Without loss of generality, the bounding polygon of B is $b_1 b_2 b_3 b_4$. In order for $r_{\{1,3\}}$ and $r_{\{2,4\}}$ to be innocent,

- $r_{\{1,3\}}$ and $r_{\{2,4\}}$ are outside of $\text{conv}(B)$;
- for $r_{\{1,3\}}$: either $b_1, b_3 \in \text{conv}(\angle b_2 r_{\{1,3\}} b_4)$ or $b_2, b_4 \in \text{conv}(\angle b_1 r_{\{1,3\}} b_3)$; and
- for $r_{\{2,4\}}$: either $b_1, b_3 \in \text{conv}(\angle b_2 r_{\{2,4\}} b_4)$ or $b_2, b_4 \in \text{conv}(\angle b_1 r_{\{2,4\}} b_3)$.

Subcase 2a: $b_1, b_3 \in \text{conv}(\angle b_2 r_{\{1,3\}} b_4)$ and $b_2, b_4 \in \text{conv}(\angle b_1 r_{\{2,4\}} b_3)$. Without loss of generality, the quadrilateral $b_4 b_1 b_2 r_{\{1,3\}}$ is convex and has b_3 inside, and without loss of generality, the quadrilateral $b_3 b_4 b_1 r_{\{2,4\}}$ is convex and has b_2 inside. Hence, $b_2 b_3 r_{\{1,3\}} r_{\{2,4\}}$ is a solid convex quadrilateral with $b_1 r_{\{2,4\}}$ outside and $b_3 r_{\{2,4\}}$ inside. Therefore, two obstacles are required, a contradiction.

Subcase 2b: $b_2, b_4 \in \text{conv}(\angle b_1 r_{\{1,3\}} b_3)$ or $b_1, b_3 \in \text{conv}(\angle b_2 r_{\{2,4\}} b_4)$. Due to symmetry, we proceed assuming the former. Without loss of generality, $Q = b_3 b_4 b_1 r_{\{1,3\}}$ is a convex quadrilateral. The obstacle is inside Q due to $r_{\{1,3\}} b_4$. In order for $b_1 r_{\{2,4\}}$ and $b_3 r_{\{2,4\}}$ to be blocked, $r_{\{2,4\}}$ is inside Q. Hence, $\angle r_{\{1,3\}} r_{\{2,4\}} b_4$ partitions $\text{conv}(Q)$ into two regions with solid boundaries that respectively contain $b_1 r_{\{2,4\}}$ and $r_{\{2,4\}} b_3$. Therefore, two obstacles are required, a contradiction.

This completes the proof of Theorem 1. □

5 Concluding remarks

A. First we answer a question from [3].

Proposition 1. *For every h, there exists a graph with obstacle number exactly h.*

Proof. Pick a graph G with obstacle number $h' > h$. (The existence of such a graph follows, e.g., from Corollary 1.) Let n denote the number of vertices of G. Consider a complete graph K_n on $V(G)$. Its obstacle number is *zero*, and G can be obtained from K_n by successively deleting edges. Observe that as we delete an edge from a graph G', its obstacle number cannot increase by more than *one*. This follows from the fact that by blocking the deleted edge with an additional small obstacle that does not intersect any other edge of G', we obtain a valid obstacle representation of the new graph. (Of course, the obstacle number of a graph can also *decrease* by the removal of an edge.) Since at the beginning of the process, K_n has obstacle number *zero*, at the end G has obstacle number $h' > h$, and whenever it increases, the increase is *one*, we can conclude that at some stage we obtain a graph with obstacle number precisely h. □

The same argument applies to the convex obstacle number, to the segment obstacle number, and many similar parameters.

B. Let H be a fixed graph. According to a classical conjecture of Erdős and Hajnal [8], any graph with n vertices that does not have an induced subgraph isomorphic to H contains an independent set or a complete subgraph of size at least $n^{\varepsilon(H)}$, for some positive constant $\varepsilon(H)$. It follows that for any hereditary graph property there exists a constant $\varepsilon > 0$ such that every graph G on n vertices with this property satisfies $\text{hom}(G) \geq n^{\varepsilon}$.

Here we show that the last statement holds for the property that the graph has bounded obstacle number.

Proposition 2. *For any fixed integer $h > 0$, every graph on n vertices with $\text{obs}_c(G) \leq h$ satisfies $\text{hom}(G) \geq \frac{1}{2} n^{\frac{1}{h+1}}$.*

Proof. We proceed by induction on h. For $h = 1$, Alpert et al. [3] showed that all graphs with convex obstacle number *one* are so-called "circular interval graphs" (intersection graphs of a collection of arcs along the circle). It is known that all such graphs G whose maximum complete subgraph is of size x has an independent set of size at least $\frac{n}{2x}$; see [29]. Setting $x = \sqrt{n/2}$, it follows that $\hom(G) \geq \frac{1}{2}\sqrt{n}$.

Let $h > 1$, and assume that the statement has already been verified for all graphs with convex obstacle number smaller than h. Let G be a graph that requires h convex obstacles, and consider one of its representations. Then we have $G = \cap_i G_i$, where G_i denotes the visibility graph of the same set of points after the removal of all but the i-th obstacle.

If the size of the largest independent set in G_1 is at least $\frac{1}{2}n^{\frac{1}{h+1}}$, then the statement holds, because this set is also an independent set in G. If this is not the case, then, by the above property of circular arc graphs, G must have a complete subgraph K of size at least $n^{\frac{h}{h+1}}$. Consider now the subgraph of $\cap_{i=2}^{h} G_i$ induced by the vertices of K. This graph requires only $h-1$ obstacles. Thus, we can apply the induction hypothesis to obtain that it has a complete subgraph or an independent set of size at least $\frac{1}{2}(n^{\frac{h}{h+1}})^{\frac{1}{h}} = \frac{1}{2}n^{\frac{1}{h+1}}$. □

It is easy to see that every graph G on n vertices with convex obstacle number at most h has the following stronger property, which implies that they satisfy the Erdős-Hajnal conjecture: There exists a constant $\varepsilon = \varepsilon(h)$ such that G contains a complete subgraph of size at least εn or two sets of size at least εn such that no edges between them belongs to G (cf. [14]).

C. Finally, we comment on higher dimensional representations. In *three* dimensions, every graph can be represented with one obstacle that is a polygonal chain.

Proposition 3. *In* three *dimensions, every* planar *graph can be represented with one convex obstacle.*

Proof. Given a planar graph G, triangulate a planar embedding of it to obtain the graph T. Now take a convex polyhedron C (no four vertices coplanar) with graph T. Let O be the convex hull of the set of midpoints of all pairs in $V(C)$ that do not correspond to edges in G. Clearly, $V(C)$ together with O (which can be perturbed to attain general position) constitute a 1-convex obstacle representation of G in *three* dimensions. □

Proposition 4. *In dimensions $d = 4$ and higher, every graph can be represented with one convex obstacle.*

Proof. Let G be a graph with n vertices. Consider the moment curve

$$\{(t, t^2, t^3, t^4) : t \in \mathbb{R}\} \ .$$

Pick n points $v_i = (t_i, t_i{}^2, t_i{}^3, t_i{}^4)$ on this curve, $i = 1, \ldots, n$. The convex hull of these points is a *cyclic polytope* P_n. The vertex set of P_n is $\{v_1, \ldots, v_n\}$, and any segment connecting a pair of vertices of P_n is an edge of P_n (lying on its boundary). Denote the midpoint of the edge $v_i v_j$ by v_{ij}, and let O be the convex hull of the set of all midpoint

v_{ij}, for which v_i and v_j are not connected by an edge in G. Obviously, the points v_i and the obstacle O (or its small perturbation, if we wish to attain general position) show that G admits a representation with a single convex obstacle. □

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